Previously, - nth Test of divergence
- Integral test - p-series
- Limits comparison test
- alternating series test (absolute convergence test)
- ✓ Ratio and Root tests

(Nice summary on page 632)

(1) Ratio Test of convergence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$$

- If $\rho < 1 \longrightarrow \sum a_n$ converges
- If $\rho > 1 \longrightarrow \sum a_n$ diverges
- If $\rho = 1 \longrightarrow$ Inconclusive!

Eg: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

Solution

$$a_n = \frac{2^n}{n!} \; ; \; a_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2\cdot 2^n}{n!(n+1)}$$

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{n}\right) = \lim_{n\to\infty}\left(\frac{2\cdot 2^n}{n!(n+1)}\cdot\frac{n!}{2^n}\right) = \lim_{n\to\infty}\left(\frac{2}{n+1}\right)$$

$$\rho = 0 < 1 \longrightarrow \sum_{n=0}^{\infty}\frac{2^n}{n!} \text{ converges!}$$

Eg: $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2}{2^n}\right)$

Let $a_n = \dfrac{n^2}{2^n}$ so $a_{n+1} = \dfrac{(n+1)^2}{2^{n+1}} = \dfrac{n^2+2n+1}{2^n\cdot 2}$

Thus, $\dfrac{a_{n+1}}{a_n} = \dfrac{n^2+2n+1}{2^n\cdot 2}\cdot\dfrac{2^n}{n^2} = \dfrac{n^2+2n+1}{2n^2}$

Now $\displaystyle\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{n^2+2n+1}{2n^2}\right) = \frac{1}{2} < 1$

$$\Longrightarrow \sum_{n=1}^{\infty}\left(\frac{n^2}{2^n}\right) \text{ converges!}$$

Eg: $\sum_{n=0}^{\infty} (-1)^n \frac{n!}{2^n}$

$a_n = (-1)^n \frac{n!}{2^n}$, $a_{n+1} = \frac{(-1)^{n+1}(n+1)!}{2^{n+1}} = \frac{(-1)(-1)^n n!(n+1)}{2^n \cdot 2}$

So $\frac{a_{n+1}}{a_n} = \frac{(-1)(-1)^n n!(n+1)}{2^n \cdot 2} \cdot \frac{2^n}{(-1)^n n!}$

$= \frac{(-1)(n+1)}{2}$

Now $\lim_{n\to\infty}\left(\left|\frac{a_{n+1}}{a_n}\right|\right) = \lim_{n\to\infty}\left(\frac{n+1}{2}\right) = \infty > 1$

So $\sum_{n=0}^{\infty} (-1)^n \frac{n!}{2^n}$ diverges!

Observe: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\qquad \sum_{n=1}^{\infty} n^2$

Let $a_n = \frac{1}{n^2} \longrightarrow a_{n+1} = \frac{1}{(n+1)^2}$

$b_n = n^2 \longrightarrow b_{n+1} = (n+1)^2$

$$\left.\begin{aligned} \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) &= \lim_{n\to\infty}\left(\frac{n^2}{n^2+2n+1}\right) = 1 \\ \lim_{n\to\infty}\left(\frac{b_{n+1}}{b_n}\right) &= \lim_{n\to\infty}\left(\frac{n^2+2n+1}{n^2}\right) = 1 \end{aligned}\right\} \longrightarrow \text{inconclusive}$$

since $\sum \frac{1}{n^2}$ converges as a $p=2$-series

while $\sum n^2$ diverges by nth-test of divergence!

(2) Root Test of convergence

$$\lim_{n\to\infty} \sqrt[n]{|a|} = L$$

if $L > 1$ then $\sum a_n$ diverges

$L < 1$ then $\sum a_n$ converges

$L = 1$ inconclusive

Eg: $$\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$$

Solution:

Let $a_n = \left(\frac{n}{2n+3}\right)^n \longrightarrow \sqrt[n]{|a|} = \frac{n}{2n+3}$

Now $\lim_{n\to\infty}\left(\sqrt[n]{|a|}\right) = \lim_{n\to\infty}\left(\frac{n}{2n+3}\right) = \frac{1}{2} < 1$

We can conclude that $\sum_{n=1}^{\infty}\left(\frac{n}{2n+3}\right)^n$ converges.

Eg: $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

Let $a_n = \frac{e^{2n}}{n^n} = \left(\frac{e^2}{n}\right)^n$, so $\sqrt[n]{|a_n|} = \frac{e^2}{n}$

Therefore $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{e^2}{n}\right) = 0 < 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$ converges!

Taylor Polynomials (9.7)

Taylor polynomial of order n is a polynomial approximation –

$$\boxed{P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(c)(x-c)^j}{j!}}$$

where c is the center of the function $f(x)$ that is being approximated

In other words:

$$f(x) \simeq P_n(x) = \frac{\overset{f(c)}{\shortparallel}\; f^{(0)}(c)(x-c)^0}{0!} + \frac{f^{(1)}(c)(x-c)^1}{1} + \frac{f^{(2)}(c)(x-c)^2}{2!} + \cdots$$

$$+ \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!}$$

When $c=0$,

$$P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)(x)^j}{j!}$$

is called the Maclaurin polynomial!

Ex: Find a Maclaurin polynomial approximation of order n of e^x.

Solution

$$f(x) = f^{(0)} = e^x$$

$$f^{(1)}(x) = e^x \quad f^{(2)}(x) = e^x \quad \ldots \quad f^{(n)}(x) = e^x$$

$$f^{(0)}(0) = 1 = \cdots\cdots = f^{(n)}(0) = e^0 = 1$$

$$P_n(x) = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

$$\boxed{e^x \approx \sum_{j=0}^{n} \frac{x^j}{j!}}$$

Ex: Find the Taylor polynomial approximation of order n, of $y = \ln x$ with center $c = 1$

$$f^{(0)}(x) = \ln x \; ; \; f^{(1)}(x) = \frac{1}{x} \; ; \; f^{(2)}(x) = -\frac{1}{x^2} \; ; \; f^{(3)} = \frac{2}{x^3} \; ; \; f^{(4)}(x) = \frac{-2\cdot 3}{x^4}$$

$$f^{(5)}(x) = \frac{+2\cdot 3\cdot 4}{x^5} \; ; \; f^{(6)}(x) = \frac{-2\cdot 3\cdot 4\cdot 5}{x^6} \; \ldots$$

Now $f^{(0)}(1) = 0$; $f^{(1)}(1) = 1$; $f^{(2)}(1) = -1$; $f^{(3)}(1) = 2$; $f^{(4)}(1) = -2.$

$f^{(5)}(1) = 2\cdot 3\cdot 4$; $f^{(6)}(1) = -2\cdot 3\cdot 4\cdot 5$

So $$P_4(x) = 0 + \frac{1(x-1)}{1!} - \frac{1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{2\cdot 3(x-1)^4}{4!}$$

$$= \frac{2\cdot 3\cdot 4(x-1)^5}{5!} - \cdots$$

$$\ln x = \sum_{j}^{n} \frac{(-1)^{j-1}(x-1)^j}{j}$$

Ex.: Find the Maclaurin polyn. approximation of $\cos x$

$f^{(0)}(0) = \cos(0) = 1$; $f^{(1)}(0) = -\sin(0) = 0$

$f^{(2)}(0) = -\cos(0) = -1$; $f^{(3)}(0) = +\sin(0) = 0$

$f^{(4)}(0) = \cos(0) = 1$; $f^{(5)}(0) = -\sin(0) = 0$

$\vdots$

$$\boxed{\cos x = \sum_{j=0}^{n} \frac{(-1)^j x^{2j}}{(2j)!}}$$

$$\frac{1x^0}{0!} + \frac{-1x^2}{2!} + \frac{1x^4}{4!} - \frac{1x^6}{6!} + \cdots$$